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Hydrodynamics and Platoon Formation for a Totally Asymmetric Exclusion Model with Particlewise Disorder

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We consider a one-dimensional totally asymmetric exclusion model with quenched random jump rates associated with the particles, and an equivalent interface growth process on the square lattice. We obtain rigorous limit theorems for the shape of the interface, the motion of a tagged particle, and the macroscopic density profile on the hydrodynamic scale. The theorems are valid under almost every realization of the disordered rates. Under suitable conditions on the distribution of jump rates the model displays a disorder-dominated low-density phase where spatial inhomogeneities develop below the hydrodynamic resolution. The macroscopic signature of the phase transition is a density discontinuity at the front of the rarefaction wave moving out of an initial step-function profile. Numerical simulations of the density fluctuations ahead of the front suggest slow convergence to the predictions of a deterministic particle model on the real line, which contains only random velocities but no temporal noise.

KEY WORDS: Interacting particle systems; quenched disorder; asymmetric exclusion; hydrodynamic limit; phase separation; traffic models.

1. INTRODUCTION

A common theme in the statistical mechanics of disordered systems is the competition between static (quenched) randomness and dynamic (thermal or nonequilibrium) fluctuations. Diverse systems such as spin glasses,[19, 22] directed polymers,[10] and driven elastic manifolds in random media[26] display sharp phase transitions between a disorder-dominated phase and a

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fluctuation-dominated phase with qualitatively distinct behaviors. There are few nontrivial cases where the mechanism underlying the transition can be analyzed in detail and with mathematical rigor. The present paper is devoted to such a system, in which a transition between disorder-dominated and fluctuation-dominated *nonequilibrium* phases occurs as a function of density.

We study a totally asymmetric disordered exclusion process introducted independently by Krug and Ferrari[14] and by Evans.[5] In this model, particles occupy sites of the 1-dimensional integer lattice **Z**. The exclusion rule admits at most one particle per site. Particles jump to the next site on the right, at exponential rates, while observing the exclusion rule: a jump to an already occupied site is suppressed. We label the particles by integers i, and then give particle i its own rate of jumping $p_i \in [c, 1]$. The rates $\{p_i\}$ are chosen at random at the outset, and then frozen for the entire dynamics. They constitute the quenched disorder, counteracted by the temporal fluctuations of the exponential waiting times.

Our paper has two parts. The first part contains rigorous limit theorems on the hydrodynamic scale. For an asymmetric particle system this means that both space and time are scaled by the same factor n. We obtain a law of large numbers for a tagged particle and for the empirical particle density. The macroscopic motion of a tagged particle is the viscosity solution of a Hamilton–Jacobi equation, while the macroscopic density profile obeys the entropy solution of a scalar conservation law. The theorems are valid under almost every realization of the disordered rates, and include both weak and strong laws of large numbers (this is the difference between convergence in probability and almost sure convergence).

The flux function of the conservation law can be expressed in terms of the probability distribution of the disorder. This is a consequence of the fact that when the disordered rates are attached to the particles, the process still has product-form steady states that can be written down explicitly. The steady state is such that the gaps between particles are mutually independent geometrically distributed random variables. The expectations of the gaps are not constant, but depend on the intrinsic rates $\{p_i\}$ in a way that makes all particles jump as Poisson processes with a common speed.

However, these product-form equilibria break down at low densities if the disorder distribution is such that unusually slow particles are rare. Then there is a critical density ρ^* such that the product equilibria exist for $\rho \geqslant \rho^*$, and not for $\rho < \rho^*$. The transition occurs when the common speed $v(\rho)$ of particles, which increases with decreasing density, becomes equal to the jump rate c of the slowest particles. Below ρ^* the slowest particles no longer interact with the particles ahead of them. Large gaps open up in front of the slow particles, and queues ("platoons") form behind them.

The system enters a disorder-dominated, inhomogeneous phase (Fig. 1). In a finite system the stationary state consists of a single platoon of density ρ^* and a macroscopic gap in front of the slowest particle. In terms of gap sizes the transition is analogous to Bose–Einstein condensation, with the gap in front of the slowest particle corresponding to the macroscopically populated ground state.[5, 6]

The inhomogeneity at $\rho < \rho^*$ is not visible on the hydrodynamic scale. For example, our theorem says that the macroscopic velocity of a tagged

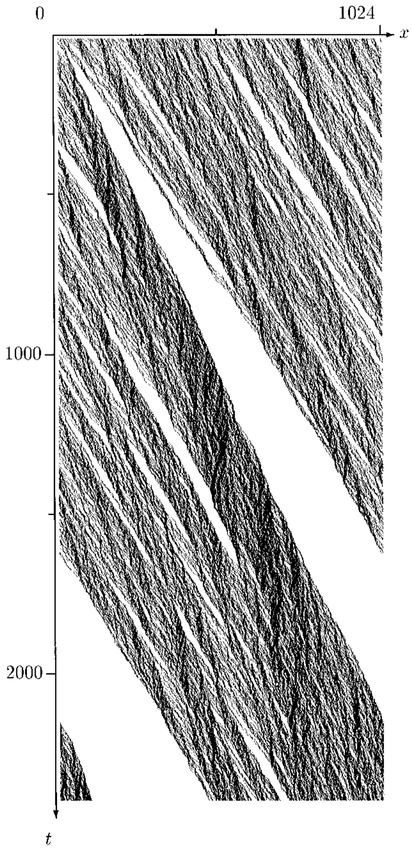

Fig. 1. Space-time plot of particle trajectories in the disordered exclusion model. Space runs from left to right, and time from top to bottom. The figure shows 256 particles on a ring of 1024 sites, with random jump rates chosen according to the disorder density (9) with $c = 0.5$ and $\nu = 2$. The initial condition was Bernoulli with density $\rho = 1/4 < \rho^* = 2/5$. Note the opening and closing of large gaps. The finite ring geometry implies a final state consisting of a macroscopic platoon of density ρ^* and a macroscopic gap; nevertheless the evolution at earlier times is characteristic of the infinite system behavior. Courtesy of M. Gerwinski.

particle is constant across the range of ambient densities $0 \leqslant \rho \leqslant \rho^*$ surrounding the particle. This suggests that, in the infinite system starting from a homogeneous initial condition, the length scale of the inhomogeneities—the typical size of platoons or gaps—grows sublinearly with time. In the second, heuristic part of the paper we recall[14, 13] a description of this length scale by means of a simplified deterministic model of platoon formation, which contains only quenched random velocities but no temporal fluctuations.[23, 3] We focus on the fluctuations in the outflow from a step-function initial condition, where all sites to the left of the origin are filled and those to the right of the origin are empty.[25] On the hydrodynamic scale the outflow is limited by a front moving at speed c, where the density jumps discontinuously from ρ^* to zero. When viewed with finer resolution the front is seen to be a source of platoons. Simulations show that these front fluctuations are described by the deterministic model of platoon formation at least asymptotically for long times.

We consider two physical interpretations of the exclusion process: as a model of an interface moving in the plane, and as a model of traffic on a single-lane highway.[14, 15, 5, 13] The role of the disorder is different in the two pictures. In the interface model the disorder introduces columnar defects. The critical density ρ^* is replaced by a critical slope u^* such that at inclinations $u \geqslant u^*$, the velocity of the interface is a constant c, while at inclinations $u < u^*$ the velocity is a strictly concave and increasing function $a(u)$ of the inclination. In the traffic model the disorder gives each vehicle its own intrinsic speed which it attempts to realize, unless blocked by the vehicles in front of it. At densities $\rho \leqslant \rho^*$, the current is linear in the density: $j(\rho) = c\rho$.

The two pictures are useful because different parts of the proofs and the discussion are more natural from one point of view than from the other. For example, the first step of our proof utilizes a last-passage formulation of a special case of the interface model, to take advantage of Kingman's subadditive ergodic theorem. This step is fundamental to our rigorous hydrodynamic limits. On the other hand, the inhomogeneities that develop on a finer scale are conveniently discussed in terms of the platoons of a traffic model.

The hydrodynamic behavior of asymmetric processes with random rates has been studied earlier by Benjamini *et al.*[2] We improve their results in several ways: Most importantly, our theorems cover the low density regime $\rho \leqslant \rho^*$ about which ref. 2 had nothing to say. Secondly, we admit arbitrary initial distributions for the process as long as a macroscopic profile exists, while ref. 2 requires local equilibrium states at the outset. And thirdly, we admit an arbitrary marginal probability distribution for the random rate, instead of the finitely supported distribution of ref. 2.

On the other hand, some assumptions of ref. 2 are less restrictive than ours. Theorem 3.1 of ref. 2, which most closely corresponds to our theorems, is valid in higher dimensions and does not require total asymmetry. Furthermore, ref. 2 admits an arbitrary ergodic process for the random rates. We assume an i.i.d. process. This can be relaxed somewhat, but not to an arbitrary process. There is a Borel–Cantelli argument at a crucial juncture in our proof, and for this we need some mixing assumption on the disorder process.

Our approach applies equally well to a discrete-time exclusion process.[6] Section 2 in ref. 28 indicates how to adapt the arguments. We expect the results to be qualitatively the same.

The exclusion process with random rates attached to the sites is considerably more difficult than the case treated here with the rates attached to the particles. The difficulty is that the equilibria of site-disordered exclusion are completely unknown. Reference 30 proves that totally asymmetric exclusion does satisfy hydrodynamic limits even with sitewise disorder. But the flux function cannot be calculated without knowledge of invariant measures, or without some new technique that we presently do not have.

The organization of our paper is as follows: Section 2 presents the rigorous hydrodynamic limits. Section 2.1 explains the limit theorem for the position of an interface, and Section 2.2 the limit theorem for the empirical density of particles. Heuristic and numerical results pertaining to the finer fluctuations are discussed in Section 3. Section 4 contains the proofs of the theorems of Section 2.

Notational Remark. For a real number x, $[x]$ denotes the largest integer $\leqslant x$.

2. THE MODELS AND THEIR HYDRODYNAMIC LIMITS

2.1. The Interface Model

Consider an interface on the two-dimensional square lattice, given by random $\mathbf{Z}$-valued height variables $h(i, t)$, where $t \geqslant 0$ is time, i is an integer site, and $h(i, t)$ is the height of the interface above site i. The interface is constrained to have nonnegative slope:

$$h(i, t) \leqslant h(i+1, t) \qquad \text{for all } i \text{ and } t \tag{1}$$

The interface moves upward through deposition events: $h(i, t)$ is increased by one at rate p_i, provided condition (1) is not violated.

The quenched disorder variables $\{p_i: i \in \mathbf{Z}\}$ are chosen i.i.d. from a distribution $f(p)\,dp$ supported on $[c, 1]$, $c > 0$. The constant c is taken to be exactly the left endpoint of the disorder distribution:

$$c = \sup\left\{b < 1 : \int_b^1 f(p)\,dp = 1\right\} \tag{2}$$

Whether or not 1 is the right endpoint of the distribution $f(p)\,dp$ is immaterial. We write $\mathbf{p} = \{p_i\}$ for the disorder configuration. The configuration $\mathbf{p}$ is an element of the space $\Pi = [c, 1]^{\mathbf{Z}}$.

Our goal is to prove that under appropriate scaling this random interface has a deterministic macroscopic limiting shape, for almost every choice of the rates $\{p_i\}$, whenever the deposition process is started from a slowly varying initial interface.

The macroscopic description involves two variables dual to each other, namely the location $x \in \mathbf{R}$ along the real line and the slope $u \geqslant 0$ of the interface. Set

$$u^* = c \int_c^1 \frac{f(p)\,dp}{p - c} \tag{3}$$

For $0 \leqslant u < u^*$, the macroscopic velocity $a = a(u)$ of the interface is defined implicitly, as a function of the slope u, by

$$u = a \int_c^1 \frac{f(p)\,dp}{p - a} \tag{4}$$

This defines a strictly increasing, strictly concave one-to-one mapping from $0 \leqslant u < u^*$ onto $0 \leqslant a < c$. If $u^* < \infty$, set

$$a(u) = c \qquad \text{for} \quad u \geqslant u^* \tag{5}$$

Equation (4) comes from an equilibrium calculation that is discussed in Sect. 2.2, see Eq. (21) below.

First we study the special case where the deposition process fills in an infinite corner. For this special case of the interface model we write $Z(i, t)$ instead of $h(i, t)$ for the process. In the beginning,

$$Z(i, 0) = 0 \quad \text{for } i \leqslant 0 \qquad \text{and} \qquad Z(i, 0) = +\infty \quad \text{for } i > 0 \tag{6}$$

Then $Z(i, t) = \infty$ for all $t \geqslant 0$ and $i > 0$. For $i \leqslant 0$, $Z(i, t)$ obeys the random dynamics, according to which $Z(i, t)$ jumps up at rate p_i, subject to $Z(i, t) \leqslant Z(i+1, t)$.

Theorem 1. Assume the initial interface is given by Eq. (6). Then for almost every choice of $\mathbf{p}$ the following holds: The limit

$$\lim_{n\to\infty} n^{-1}Z([nx], nt) = tg(x/t) \tag{7}$$

holds for all $x<0$ and $t>0$, almost surely with respect to the random deposition dynamics. The limit function $g(x)$ is nonrandom, convex, and continuous, and it is the Legendre conjugate of the (negative of the) velocity function:

$$g(x) = \sup_{u\geqslant 0}\{xu + a(u)\}, \qquad x<0 \tag{8}$$

The almost sure convergence of the theorem refers to a particular construction of the interface process $Z(i, t)$ in terms of a last-passage model, which we undertake in Sect. 4.1. The theorem says that for typical $\mathbf{p}$, the limiting shape g is independent of the particular realization of $\mathbf{p}$, and is determined by the distribution $f(p)\,dp$ through relations (4)–(5) and (8). The limit in Eq. (7) for $x=0$ is determined by the rate p_0, so we only consider $x<0$ to get the averaging effect on the disorder.

Figure 2 shows two simulations of $Z(i, t)$ at time $t=5000$. The disorder density on $[c, 1]$ is

$$f(p) = \frac{\nu+1}{(1-c)^{\nu+1}}(p-c)^{\nu} \tag{9}$$

with values $\nu=0$ and $\nu=2$ for the two interfaces. See Sect. 3.3 for further discussion of the pictures.

Properties of *g*. First we have

$$g(x)=0 \quad \text{for } x\leqslant -a'(0) \quad \text{and} \quad g(x)>0 \quad \text{for } -a'(0)<x<0 \tag{10}$$

where $a'(u)$ is, by Eq. (4),

$$a'(u) = \left\{\int_c^1 \frac{pf(p)\,dp}{(p-a(u))^2}\right\}^{-1}, \qquad 0\leqslant u<u^* \tag{11}$$

The derivative $g'(x)$ is continuous for $-\infty<x<0$. At the right endpoint of the graph of g we have the value $g(0-)=c$, so we can make g continuous on $(-\infty, 0]$ be defining $g(0)=c$. The slope and curvature of the

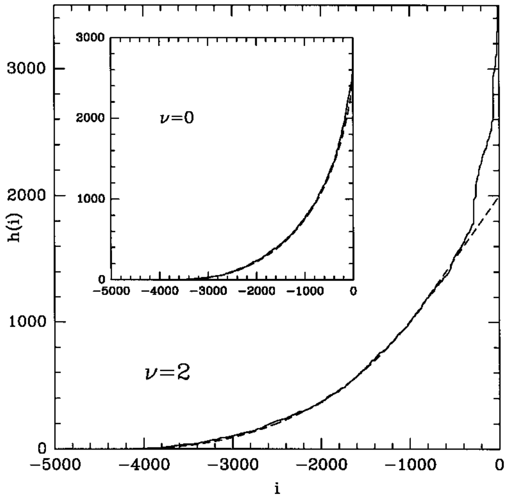

Fig. 2. Simulation of the random interface growth process $Z(i, t)$ filling an infinite corner. Full lines are single realizations at time $t = 5000$. The actual simulation was carried out on a lattice of 5000 sites, but it was ensured that the growth process did not reach the edges of the lattice. The main figure shows an interface generated using the disorder density (9) with $c = 0.4$ and $v = 2$, while the inset shows the case $c = 0.5$, $v = 0$. The dashed lines depict the predicted deterministic growth shape, and were computed from Eqs. (4) and (8).

approach to this value depend on the tail of the distribution $f(p)$ as $p \searrow c$. Let

$$x^* = -a'(u^* -) \tag{12}$$

By the duality Eq. (8), $g'(x^* -) = u^*$. Under the duality, a jump in the derivative $a'(u)$ corresponds to a linear segment in g. Depending on the value of u^* and the behavior of $a'(u)$ at u^*, the following three cases emerge:

Case I:

$$\int_c^1 \frac{f(p)\,dp}{p-c} = \infty$$

Then $u^* = \infty$, $x^* = 0$, $g(x)$ is strictly convex for $-a'(0) < x < 0$, and $g'(0-) = \infty$.

Case II:

$$\int_c^1 \frac{f(p)\,dp}{p-c} < \infty = \int_c^1 \frac{f(p)\,dp}{(p-c)^2}$$

Then still $x^* = 0$ and $g(x)$ is strictly convex for $-a'(0) < x < 0$, but $g'(0-) = u^* < \infty$, so the interface has a corner at $x = 0$ where it attaches to the infinite vertical wall.

Case III:

$$\int_c^1 \frac{f(p)\,dp}{(p-c)^2} < \infty$$

Now $u^* < \infty$ and $x^* < 0$. $g(x)$ is strictly convex for $-a'(0) < x < x^*$, and has a linear segment of slope u^* for $x^* \leqslant x \leqslant 0$. Again, there is a corner at $x = 0$.

Next we discuss the hydrodynamic limit for more general initial interfaces. The initial macroscopic interface is an arbitrary nondecreasing function $U_0(x)$ on $\mathbf{R}$, normalized to be continuous from the right. The initial microscopic interfaces are given by a sequence $\{h_n(i, 0)\colon i \in \mathbf{Z}\}$, $n = 1, 2, 3,...,$ of random initial interfaces, each satisfying (1). The assumption that connects the microscopic initial interfaces with the macroscopic initial interface is that

$$\lim_{n \to \infty} n^{-1} h_n([nx], 0) = U_0(x) \qquad \text{for} \quad x \in \mathbf{R} \tag{13}$$

Pick a realization $\mathbf{p} = \{p_i\}$ of the rates, independently with common marginal $f(p_i)\,dp_i$. Denote by $h_n(i, t)$ the process with initial interface $h_n(i, 0)$ that evolves with rates $\{p_i\}$, as described in the paragraph around Eq. (1). The point of the setup is this: As the parameter n (the ratio between the microscopic and macroscopic scale for both space and time) tends to infinity, the random microscopic initial interface $h_n([nx], 0)$ closely approximates the macroscopic, non-random interface $U_0(x)$, according to Assumption (13). We let the microscopic interface evolve according to its stochastic dynamics. We prove that at later macroscopic times $t > 0$, the microscopic interface $h_n([nx], nt)$ again closely approximates a macroscopic, non-random interface $U(x, t)$, in the limit as $n \to \infty$.

We shall consider two mathematically precise versions of assumption (13). In both cases all the initial interfaces $h_n(\cdot, 0)$ and the disorder $\mathbf{p}$ are defined jointly on a probability space $(\Omega, \mathscr{F}, P)$. The marginal distribution on $\mathbf{p}$ is the i.i.d. $f(p)\,dp$-distribution. Given a realization $\mathbf{p}$ of the rates, the initial distribution for the nth process h_n is the conditional distribution $P(dh_n(\cdot, 0) \mid \mathbf{p})$. If the joint distribution of the initial interfaces and the disorder is not specified, we can take them to be independent. It is important to allow the initial distribution $P(dh_n(\cdot, 0) \mid \mathbf{p})$ of the process to depend on $\mathbf{p}$. Otherwise we cannot work with the equilibria of the process that do depend on $\mathbf{p}$, as will be seen in the next section.

Assumption A. The convergence in Eq. (13) holds simultaneously for all $x \in \mathbf{R}$, P-almost surely. By the monotonicity of U_0 and $h_n(i, 0)$, this is equivalent to the apparently weaker assumption that for each fixed $x \in \mathbf{R}$, Eq. (13) holds P-almost surely. It is even sufficient to require this for only a countable set of $x \in \mathbf{R}$, provided these x's are dense in $\mathbf{R}$ and include the discontinuities of U_0.

Assumption B. There is a full-measure set Π_0 of disorder configurations $\mathbf{p}$ such that the convergence in Eq. (13) holds in $P(\cdot \mid \mathbf{p})$-probability, for all $\mathbf{p} \in \Pi_0$ and all $x \in \mathbf{R}$. Again, it is equivalent to have full-measure sets $\Pi_{0,x}$ depending on x so that Eq. (13) holds in $P(\cdot \mid \mathbf{p})$-probability for $\mathbf{p} \in \Pi_{0,x}$, as long as these x's form a dense set that contains the discontinuities of U_0. However, it is not sufficient to require the convergence (13) in P-probability only.

The probability space $(\Omega, \mathscr{F}, P)$ also supports the Poisson clocks of the graphical representation that defines the dynamics. The construction is such that, given $\mathbf{p}$, the initial interfaces $h_n(i, 0)$ and the Poisson clocks are independent. [This independence cannot hold unconditionally because $\mathbf{p}$ gives the rates of the Poisson clocks and may influence the distribution of the initial interface, as stressed above.] The interface processes $h_n(i, t)$ are defined on this same probability space as functions of the initial interfaces $h_n(i, 0)$ and the Poisson clocks. See Sect. 4.2 for details of the construction. The probability measure P gives annealed probabilities, which means that the disorder is averaged out. Quenched probabilities come from the conditional measures $P(\cdot \mid \mathbf{p})$ that correspond to looking at the evolution under one fixed choice of rates $\mathbf{p}$.

Theorem 2. There is a full-measure subset $\Pi_1 \subseteq \Pi$ of disorder configurations such that whenever $\mathbf{p} \in \Pi_1$ the following holds: Under Assumption A the limit

$$\lim_{n \to \infty} n^{-1} h_n([nx], nt) = U(x, t) \tag{14}$$

holds simultaneously for all $x \in \mathbf{R}$ and $t > 0$, almost surely under the measure $P(\cdot \mid \mathbf{p})$. Under Assumption B the limit holds in $P(\cdot \mid \mathbf{p})$-probability for each fixed (x, t).

The limit $U(x, t)$ is macroscopically defined by

$$U(x, t) = \inf_{y:\, y \geqslant x} \left\{ U_0(y) + tg\left(\frac{x - y}{t}\right) \right\} \tag{15}$$

or, equivalently, as the unique viscosity solution of the Hamilton–Jacobi equation

$$\frac{\partial U}{\partial t} = a\left(\frac{\partial U}{\partial x}\right), \qquad U(x,0) = U_0(x) \tag{16}$$

Remarks.

(i) The almost sure convergence obtained under Assumption A in the theorem is true for our particular construction of the interface process, in terms of a graphical representation for a disordered exclusion process (Sect. 4.2). The convergence in probability under Assumption B is also proved from this same construction. But convergence in probability is valid for any construction of the interface process because the probabilities are always the same.

The set Π_1 of "typical" disorder configurations $\mathbf{p}$, whose existence the theorem asserts, is defined in a way that is inherently tied together with the initial interfaces and the construction of the dynamics. [See Lemma 8 and the paragraphs preceding equations (94) and (98) in Sect. 4.5.] Consequently we do not assert that a single set Π_1 works for all situations.

(ii) Even though we have written $f(p)\,dp$ for the marginal disorder distribution as if to suggest that it has to be absolutely continuous, this is not the case. Any distribution on $[c, 1]$ is acceptable. However, as Cases II and III indicate, the interesting case is the one where there is no point mass at c, but instead a thin tail as $p \searrow c$. In Sect. 3.2 we assume that $f(p)\,dp$ has no point masses in order to have $p_i \neq p_j$ for $i \neq j$ almost surely.

The assumption that $\{p_i\}$ are i.i.d. can also be relaxed. For Theorem 1 no assumption beyond ergodicity is needed on $\{p_i\}$. For Theorem 2 we need an application of the Borel–Cantelli lemma (see Lemma 8 in Sect. 4.5), which in turn needs summable probability estimates that we prove in Sect. 4.4.

(iii) That formula (15) defines the unique viscosity solution of the initial value problem (16) is a standard fact from the theory of Hamilton–Jacobi equations.[1, 18] Our proof is based on Eq. (15) and makes no use of the partial differential equation (16).

(iv) Equation (16) says that the function $a(u)$ defined by Eqs. (4)–(5) is indeed the macroscopic velocity of the interface. In particular, suppose the initial interface has a slope that may vary from point to point but is still $\geqslant u^*$ everywhere. Then Eqs. (14), (16), and (5) imply that in the hydrodynamic scale this interface moves rigidly upward with constant velocity c.

(v) The reader may be puzzled by the following apparent contradiction: According to conclusion (14) of the theorem, the limiting velocity of $h_n(0, nt)$ is not influenced by its intrinsic rate p_0. But what if the interface takes an upward step of order n at the point zero, so that

$$h_n(1, 0) = h_n(0, 0) + \varepsilon n \tag{17}$$

Then $h_n(0, nt)$ is not held back by $h_n(1, nt)$ for a time of order n. Consequently $h_n(0, nt)$ moves ahead at its intrinsic rate p_0 for a macroscopic amount of time. This ought to be visible in the limit (14), at least for small enough $t > 0$. How can this be resolved?

The answer lies in the assumption of right continuity of U_0, made in the paragraph before Eq. (13). If Eq. (17) were true for all large n, then $U_0(0+) \geqslant U_0(0) + \varepsilon$ by Eq. (13), contradicting right continuity.

2.2. The Exclusion Model

The interface motion can be represented in terms of the totally asymmetric simple exclusion process (TASEP). Write $\sigma(i, t)$ for the location of particle i at time t on the integer lattice $\mathbf{Z}$. The requirement is that

$$\sigma(i, t) + 1 \leqslant \sigma(i+1, t) \tag{18}$$

always. This contains the exclusion rule which stipulates that each site contains at most one particle, and also an ordering convention which is preserved by the dynamics. Particle i jumps one step to the right at rate p_i, provided the next site is vacant. The connection with the interface is that

$$\sigma(i, t) = h(i, t) + i \tag{19}$$

The slope u of the interface now corresponds to the gap (= number of empty sites) between successive particles. A more natural variable in the particle context is the density ρ, given in terms of u by $\rho = (1+u)^{-1}$. Instead of $a(u)$ we now write $v(\rho)$ for the velocity, so that $v(\rho) = a(u) = a((1-\rho)/\rho)$.

The critical slope u^* of the interface becomes a critical density $\rho^* = (1+u^*)^{-1}$ of the disordered TASEP. For low density $\rho \leqslant \rho^*$, $v(\rho) = c$. For high density $\rho > \rho^*$ the velocity $v(\rho)$ can be calculated from explicit product-form equilibria: For a fixed $v < c$ and a choice $\{p_i\}$ of rates, we give the gaps

$$\eta(i) = \sigma(i+1) - \sigma(i) - 1$$

the following independent geometric distributions:

$$P[\eta(i)=k]=(1-v/p_i)(v/p_i)^k, \qquad k=0, 1, 2, 3,... \tag{20}$$

Since gap $\eta(i)$ can be regarded as an M/M/1 queue with service rate p_i, this equilibrium is preserved by the dynamics, and each particle jumps according to a Poisson(v) process. (See ref. 11 for an explanation of this property of M/M/1 queues in series.) The quenched average gap is

$$\langle \eta(i) \rangle = \sum_{k=0}^{\infty} k(1-v/p_i)(v/p_i)^k = \frac{v}{p_i - v}$$

Averaging this over $f(p_i)\, dp_i$ gives the following equation that links ρ, u, and v:

$$\frac{1-\rho}{\rho} = u = \overline{\langle \eta(i) \rangle} = v \int_c^1 \frac{f(p)\, dp}{p-v} \tag{21}$$

The overbar on $\overline{\langle \eta(i) \rangle}$ indicates an averaging over the disorder. Equation (21) defines $v = v(\rho)$ for $\rho \in (\rho^*, 1]$. This equilibrium calculation is the justification of Eq. (4) referred to earlier.

The product-form equilibria break down at a nonzero density $\rho^* > 0$ if the disorder density belongs to one of the Cases II or III in 2.1. In Case II the phase transition at $\rho^* > 0$ is of *second order*, in the sense that the first derivative of the tagged particle velocity $v(\rho)$ is continuous at $\rho = \rho^*$, while in Case III the transition is of first order (discontinuous first derivative).[14]

Using Eq. (19) Theorem 2 can be restated as a law of large numbers for a tagged particle in the disordered TASEP. As a further corollary we state a hydrodynamic scaling limit for the conserved empirical particle density. Suppose we have a sequence $\sigma_n(i, t)$, $n = 1, 2, 3,...,$ of processes that initially satisfy

$$\lim_{n \to \infty} n^{-1} \sigma_n([nx], 0) = V_0(x) \tag{22}$$

in $P(\cdot \mid \mathbf{p})$-probability, for all $x \in \mathbf{R}$ and all typical $\mathbf{p}$. By Eq. (18) the function V_0 must have slope at least 1 everywhere, and therefore a continuous, nondecreasing inverse V_0^{-1} exists. The derivative

$$\rho_0(x) = (d/dx)\, V_0^{-1}(x) \tag{23}$$

exists almost everywhere, and gives the initial macroscopic particle density. The empirical particle density is a random measure $\mu_n(t, dx)$ on $\mathbf{R}$, defined through the integrals

$$\mu_n(t, \phi) = n^{-1} \sum_{i \in \mathbf{Z}} \phi(n^{-1}\sigma_n(i, t))$$

of compactly supported test functions ϕ.

Theorem 3. Assume Eq. (22). Then the limit

$$\lim_{n \to \infty} \mu_n(nt, dx) = \rho(x, t)\, dx \tag{24}$$

holds in $P(\cdot \mid \mathbf{p})$-probability, for all typical disorder configurations $\mathbf{p}$. The nonrandom limit density $\rho(x, t)$ is the entropy solution of the conservation law

$$\rho_t + [\rho v(\rho)]_x = 0, \qquad \rho(\cdot, 0) = \rho_0 \tag{25}$$

where the initial density ρ_0 is defined by Eq. (23).

This theorem corresponds to Theorem 2 under Assumption B, and there is of course a version that corresponds to Assumption A also. The convergence of measures in Eq. (24) means convergence of all integrals against continuous, compactly supported test functions:

$$\lim_{n \to \infty} \mu_n(nt, \phi) = \int_{\mathbf{R}} \phi(x)\, \rho(x, t)\, dx$$

That $\rho(x, t)$ is the entropy solution of Eq. (25) means this: Let $R_0 = V_0^{-1}$ so that $R_0'(x) = \rho_0(x)$. Then define $R(x, t)$ by

$$R(x, t) = \sup_y \left\{ R_0(y) + tj^* \left(\frac{x - y}{t} \right) \right\} \tag{26}$$

where j^* is the conjugate of the current $j(\rho) = \rho v(\rho)$ of Eq. (25):

$$j^*(x) = \inf_{0 \leqslant \rho \leqslant 1} \{x\rho - j(\rho)\}$$

Finally define $\rho(x, t) = (\partial/\partial x)\, R(x, t)$. (See ref. 16 for a discussion of formula (26) in the context of conservation laws.)

Note that the duality equation for convex functions requires a supremum, while concave functions need an infimum. Compare Eqs. (8) and (27).

3. BEYOND THE HYDRODYNAMIC SCALE

Suppose the exclusion process $\sigma(i, t)$ starts in an initial configuration with density $\rho < \rho^*$ everywhere. Theorem 3 then asserts that on the hydrodynamic scale the initial density profile is simply translated at speed c, without changing its shape. [Recall that c is the left endpoint of the disorder distribution, defined by Eq. (2).] Simulations show, however, that when viewed with finer resolution, regions of density $\rho < \rho^*$ become strongly inhomogeneous[14, 15] (see Fig. 1). More precisely, a length scale $\xi(t)$ can be introduced such that on scales smaller than ξ the density profile is a superposition of empty regions ($\rho = 0$) and regions of density $\rho = \rho^*$. The underlying mechanism is the formation of *platoons*, queues which accumulate behind particles with exceptionally low jump rates. Within the platoons the density is ρ^*, while in the gaps between platoons $\rho = 0$.

Theorem 3 implies that $\xi(t)$ grows sublinearly in time,

$$\lim_{t \to \infty} \xi(t)/t = 0 \tag{28}$$

In ref. 14 a description of the platoons in terms of a deterministic particle model on $\mathbf{R}$ was proposed, which leads to the power law behavior

$$\xi(t) \sim t^{(\nu+1)/(\nu+2)} \tag{29}$$

Here ν is an exponent characterizing the tail of the disorder distribution $f(p)$ as $p \searrow c$,

$$\nu = \lim_{p \searrow c} \frac{\log f(p)}{\log(p-c)} \tag{30}$$

Cases I–III introduced in Sect. 2.1 correspond to $\nu \leqslant 0$ (Case I), $0 < \nu \leqslant 1$ (Case II), and $\nu > 1$ (Case III). The existence of the platoon phase, i.e., $\rho^* > 0$, requires $\nu > 0$. According to Eq. (29) the length scale $\xi(t)$ therefore grows faster than diffusively, which can be interpreted as a requirement for the stability of the platoons against the noise in the particle motion.[13]

There is some numerical evidence for the validity of Eq. (29) for disordered exclusion processes.[15] But also significant deviations have been reported.[13, 14] Here our purpose is to clarify this situation by focusing on the density fluctuations which occur in the outflow from a step-function initial density profile (a "megajam"). Simulations of the outflow are compared with a deterministic model of platoon formation, the *Newell model*.[3, 23] To this end we first describe the hydrodynamic density profile of the outflow, as obtained in Theorem 1 (Sect. 3.1), then discuss the Newell model (Sect. 3.2), and finally compare the predictions to numerical simulations (Sect. 3.3).

3.1. Outflow from a Megajam

In the traffic model the initial condition of Theorem 1 corresponds to a megajam,[20, 21] i.e. a semi-infinite train of particles $i=0, -1, -2, -3,...$ with initial positions

$$\sigma(i, 0)=i, \qquad i \leqslant 0 \tag{31}$$

This configuration corresponds to the step-function initial macroscopic density profile

$$\rho_0(x)=\begin{cases} 1, & x \leqslant 0 \\ 0, & x>0 \end{cases} \tag{32}$$

For $t>0$ the density profile $\rho(x, t)$ that follows from Theorem 1 or Theorem 3 is given by

$$\rho(x, t)=[1+g'(w/t)]^{-1} \tag{33}$$

where $w=w(x, t)$ is determined by

$$x=w+tg(w/t) \tag{34}$$

This can be written in the form

$$\rho(x, t)=r(x/t) \tag{35}$$

where $r(y)=0$ for $y>c$ and $r(y)=1$ for $y<-c_1$ with

$$c_1=-v'(1-)=\left[\int_c^1 p^{-1}f(p)\, dp\right]^{-1} \tag{36}$$

The interval $[-c_1 t, ct]$ contains the rarefaction wave in which the density decreases from $\rho=1$ to $\rho=0$. In the case of interest here, where $\rho^*>0$ (Cases II and III of Sect. 2.1), $r(y)$ is discontinuous at the front position $y=c$, with $r(c-)=\rho^*$ and $r(c+)=0$. In Case II $r(y)$ is strictly decreasing and satisfies

$$j'(r(y))=y \tag{37}$$

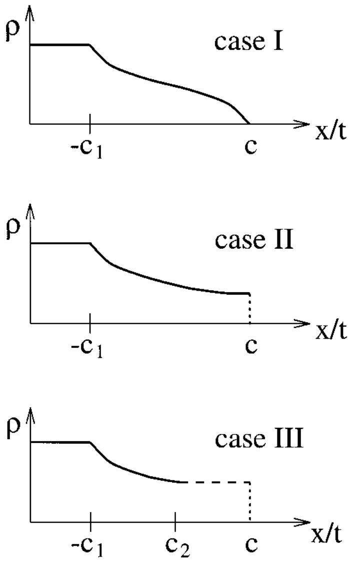

Fig. 3. Schematic of the hydrodynamic rarefaction profile for the Cases I, II, and III of the disorder density. The vertical dotted line at $x/t = c$ indicates the density discontinuity at the front, and the horizontal dashed line is a segment of constant density $\rho = \rho^*$.

for $y \in (-c_1, c)$. In Case III $r(y)$ is constant and equal to ρ^* in the interval (c_2, c), where

$$c_2 = j'(\rho^* +) = c + \rho^* v'(\rho^* +) < c \tag{38}$$

and obeys (37) in $(-c_1, c_2)$. The three cases are illustrated schematically in Fig. 3.

3.2. The Newell Model

In the Newell model[23] each particle i is assigned a position $x_i(t) \in \mathbf{R}$ and an intrinsic speed $p_i \in [c, 1]$ chosen at random from a continuous distribution $f(p)$. The particle moves at speed p_i as long as the headway to the particle ahead exceeds some threshold value Δ, and otherwise adopts the speed of the next particle ahead. In other words,

$$\frac{dx_i}{dt} = v_i = \begin{cases} p_i, & x_{i+1} - x_i > \Delta \\ \min[p_i, v_{i+1}], & x_{i+1} - x_i = \Delta \end{cases} \tag{39}$$

Ben-Naim, Krapivsky, and Redner[3] set $\Delta = 0$, which means that particles coalesce upon contact.

Here we propose using the Newell model to describe the fluctuations in the vicinity of the front of the rarefaction wave, moving at speed c, where (on the hydrodynamic scale) the density equals ρ^*. The initial condition for the particle positions will therefore be chosen as

$$x_i(0) = i\,\Delta, \qquad i = 0, -1, -2,... \tag{40}$$

with

$$\Delta = 1/\rho^* \tag{41}$$

The first particle then moves at its intrinsic speed, $v_0 = p_0$, while the speeds of the following particles are determined recursively from (39). This yields

$$v_i = \min_{j:\, i \leqslant j \leqslant 0} \{p_j\} \tag{42}$$

As a simple application we ask for the expected number N_p of platoons formed by the first N particles. A given particle i will be heading a platoon iff $p_i = v_i = \min_{i \leqslant j \leqslant 0} \{p_j\}$. Assuming that the common distribution of the speeds is continuous, the probability for this to be true is $1/i$ by symmetry.[7] Therefore

$$N_p = \sum_{i=1}^{N} \frac{1}{i} \approx \ln N + C \tag{43}$$

where C is Euler's constant. Similarly the distribution of v_i follows from Eq. (42) by elementary order statistics.[7] Using the density (9), the disorder-averaged speed for large $|i|$ becomes

$$\overline{v_i} = c + \frac{1-c}{v+1}\,\Gamma(1/(v+1))\,|i|^{-1/(v+1)} \tag{44}$$

We have seen above that on the hydrodynamic scale $\rho(x, t) = 0$ for $x > ct$. It is therefore of interest to ask for the number $\mathcal{N}(t)$ of particles ahead of the rarefaction front $x = ct$. Within the Newell model the distribution of $\mathcal{N}$ is easily calculated.[13] We first observe that the first N particles have moved beyond the front at time t iff the slowest among them has traveled a distance $N\Delta + ct$ from its starting point. By definition this particle is the head of a platoon, and therefore its speed is

$$v_{\min}(N) = p_{\min}(N) = \min_{-N \leqslant i \leqslant 0} \{p_i\} \tag{45}$$

It follows that

$$\begin{aligned}\mathrm{Prob}[\mathcal{N}(t)>N] &= \mathrm{Prob}[p_{\min}(N)>N\Delta/t+c] \\ &= \left[1-\left(\frac{N\Delta}{(1-c)\,t}\right)^{\nu+1}\right]^{N} \approx \exp[-N^{\nu+2}/\tau^{\nu+1}]\end{aligned} \tag{46}$$

where we have used the distribution (9) and assumed large N and

$$\tau=(1-c)\,t/\Delta=(1-c)\,\rho^{*}t\gg 1 \tag{47}$$

By differentiating Eq. (46) with respect to N, we obtain the probability density for $\mathcal{N}$:

$$P_t(\mathcal{N})=(\nu+2)(\mathcal{N}/\tau)^{\nu+1}\exp[-\mathcal{N}^{\nu+2}/\tau^{\nu+1}] \tag{48}$$

Its first moment equals

$$\bar{\mathcal{N}}(t)=\Gamma\left(\frac{\nu+3}{\nu+2}\right)[(1-c)\,\rho^{*}t]^{(\nu+1)/(\nu+2)} \tag{49}$$

which is proportional to the conjectured platoon scale (29). Within the Newell model the typical gap between particles has been shown analytically to scale as (29).[3] This suggests that $\bar{\mathcal{N}}(t)$ provides a measure of the platoon size also in the disordered exclusion model. In the next section we therefore present simulation results for this quantity.

3.3. Simulations

In Fig. 2 we show numerically generated realizations of the growth process $Z(i,t)$ defined in Sect. 2.1, at time $t=5000$. In these simulations the continuous time stochastic process is realized in the standard way: A column i is selected at random, and provided $h(i,t)\leqslant h(i+1,t)-1$, the height at i is increased with probability p_i. The two interfaces shown in the figure correspond to $\nu=0$ and $\nu=2$, respectively. While for $\nu=0$ the data are essentially indistinguishable from the deterministic shape, for $\nu=2$ strong deviations occur near the edge at $i=0$. In the particle picture the "excess mass" above the deterministic shape corresponds exactly to those particles which escape from the front of the rarefaction wave.

For a quantitative characterization of this effect we turn to the number $\mathcal{N}$ of such particles introduced in Sect. 3.2. Because of the equivalence

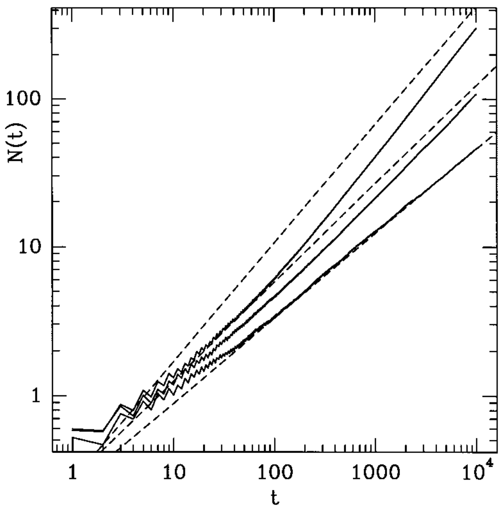

Fig. 4. Simulation of the disordered exclusion process with a step function initial density profile. The simulation was carried out on a finite lattice of 10000 sites, but it was ensured that the rarefaction wave had not reached the edges of the lattice at the end of the simulation ($t = 10^4$). The full lines show the average number of particles ahead of the rarefaction front, $\bar{\mathcal{N}}(t)$, obtained by averaging over 200 realizations of the disorder density (9). In all cases $c = 0.5$ and rom top to bottom the curves correspond to $\nu = 3$, $\nu = 1$ and $\nu = 1/3$. The dashed lines are the prediction (49) of the Newell model.

between the interface model and the exclusion process, the simulation can be carried out using the interface representation. Figure 4 shows simulation results for the average of $\mathcal{N}$, obtained from 200 independent runs. The figure indicates convergence to the Newell model prediction (49), depicted by dashed lines, but the rate of convergence is slow, particularly for large ν. Moreover it can be seen that the numerical data approach the prediction (49) from below for $\nu \geqslant 1$, but from above for $\nu = 1/3$. As a consequence the effective scaling exponent $1/z$ estimated from a power law fit

$$\bar{\mathcal{N}}(t) \sim t^{1/z} \tag{50}$$

to the numerical data tends to exceed the prediction $1/z = (\nu + 1)/(\nu + 2)$ for $\nu > 1$, but lies below this value for $\nu < 1$. For example, a fit to the data for $\nu = 3$ in the time interval $50 \leqslant t \leqslant 5000$ yields $1/z = 0.86 \pm 0.05$ instead of the predicted value 4/5.

Note that to be able to conclude that this deviation will eventually disappear it is crucial to know the prefactor in the predicted power law (49), because otherwise one cannot decide whether the data approach the prediction or diverge from it. Thus the main advantage of using the megajam geometry and the quantity $\mathcal{N}(t)$ as a characterization of the platoon scale is that the full distribution of $\mathcal{N}$ is easily computed within the Newell model.

Similar but more pronounced systematic discrepancies between numerically measured and predicted scaling exponents were observed in ref. 14, where the platoon scale $\xi(t)$ was estimated from the variance of the particle headways; in particular, the effective scaling exponent $1/z$ was larger than $(v+1)/(v+2)$ for $v>1$ but smaller for $v<1$. The present results suggest that this is a transient effect, and that asymptotically also the quantity considered in ref. 14 will follow the power law (29).

4. PROOFS

The proofs of the theorems of Sect. 2 are presented in the following order. We start with the existence of the limit in (7) and derive some properties of the limit function g [Sect. 4.1]. This proof uses a completely different construction of the interface process than the later proof of Theorem 2. In Sect. 4.2 we define the exclusion process (TASEP) in terms of a graphical construction, and develop the central technical tool of the paper, namely a coupling of TASEP with a version of $Z(i, t)$ defined by the graphical construction. After this we start working towards the proof of Theorem 2. We represent the interface by the TASEP according to Eq. (19). The proof needs the duality formula (8), so in Sect. 4.3 we apply the coupling of Sect. 4.2 to derive (8). This completes the proof of Theorem 1. In Sect. 4.3 we also justify those properties of g that were not proved earlier in Sect. 4.1. Sect. 4.4 proves a large deviation estimate for $Z(i, t)$. The argument utilizes both constructions of $Z(i, t)$, as well as the duality (8) and the equilibria (20) of the disordered TASEP. After these preliminaries, the proof of Theorem 2 follows in Sect. 4.5. Finally, Sect. 4.6 contains the derivation of Theorem 3 from Theorem 2.

When we talk simultaneously about several different processes denoted by different Greek letters, it becomes convenient to use the location variable also as the name of a particle: For example, "particle $\sigma(i)$" instead of the longer "particle i of the σ-process."

Throughout, c is the constant defined in Eq. (2). C, C_1, and C' denote constants whose exact values are immaterial and may change from line to line.

4.1. Existence of the Limit in Theorem 1

In this section we prove that $n^{-1}Z([nx], nt)$ converges to a limit $tg(x/t)$ and derive some properties of g. Identification of g as the Legendre conjugate of $-a(u)$ comes later in Sect. 4.3.

In the special setting of Theorem 1 we can define the moving interface problem as a last-passage problem and obtain the existence of the limit essentially for free from Kingman's subadditive ergodic theorem.[12] Let $\mathbf{v} = \{v_{i,j} : i \leqslant 0, j \leqslant 1\}$ be a collection of i.i.d. random variables, exponentially distributed with rate 1, independent of the disorder $\mathbf{p}$. Write Q for the joint distribution of $(\mathbf{p}, \mathbf{v})$. Q is a product measure with the appropriate marginals on $\mathbf{p}$ and $\mathbf{v}$. The quantity $p_i^{-1} v_{i,j}$ is the passage time of site (i, j). Under the conditional measure $Q(\cdot \mid \mathbf{p})$ site (i, j) has an exponential passage time of rate p_i. This is the quenched setting where we regard the p_i's fixed and the $v_{i,j}$'s random.

The time $T(i, j)$ when site (i, j) joins the growing cluster is defined by

$$T(i, j) = \max_{\pi} \sum_{(m, n) \in \pi} p_m^{-1} v_{m,n} \tag{51}$$

for $i \leqslant 0$, $j \geqslant 1$. Here π stands for a lattice path that connects $(0,1)$ to (i, j), and is directed along the negative i- and positive j-directions. With a reinterpretation of waiting times as (negative) energies, Eq. (51) defines a problem of zero temperature directed polymers or optimal path on a random lattice with columnar disorder.[10, 14] According to Eq. (6), we can think that $T(i, 0) = 0$ for $i \leqslant 0$. We define the height variables by

$$Z(i, t) = \min\{ j \geqslant 0 : T(i, j+1) > t\} \tag{52}$$

In the next section we introduce another construction of the interface process $Z(i, t)$, less suited for proving the existence of the limit in Theorem 1, but more directly connected with the exclusion process.

The important fact is that the coordinate process $\{p_i, v_{i,j}\}$ is ergodic under translations. The subadditive ergodic theorem, together with the arguments on pp. 563–564 of ref. 8, can be used to show that, for $x < 0$ and $y > 0$, there is a finite, deterministic, continuous, concave function $\gamma(x, y)$ such that

$$\lim_{n \to \infty} n^{-1} T([nx], [ny]) = \gamma(x, y) \tag{53}$$

Q-almost surely and in $L^1(Q)$. In fact, this convergence (53) holds for all (x, y) outside a single Q-null set of exceptional $(\mathbf{p}, \mathbf{v})$. To achieve this, first define $\gamma(x, y)$ for rational (x, y) through the limit (53). Then observe that the function $\gamma(x, y)$ is concave and continuous, and thus extends to all $\{x < 0, y > 0\}$ as a concave, continuous function. By the countability of rationals, there is a single null set outside of which the convergence (53) holds for all rational (x, y). Outside this same null set the convergence extends to all $\{x < 0, y > 0\}$ by virtue of the continuity of $\gamma(x, y)$.

To get the existence of the limit in Eq. (7), we need two more observations, namely that $\gamma(x, y)$ is strictly monotone in both variables, and homogeneous: $\gamma(rx, ry) = r\gamma(x, y)$ for $r > 0$. Then Eq. (7) follows for $g(x)$ defined by

$$g(x) = \inf\{ y > 0 : \gamma(x, y) \geqslant 1 \}, \qquad x < 0 \tag{54}$$

To conclude this section we deduce some basic properties of g.

From the definition of g and the properties of γ follows that g is nonnegative, nondecreasing, continuous, and convex. Since

$$E^Q[p_i^{-1} v_{i,j}] = \int_c^1 p^{-1} f(p)\, dp = \frac{1}{a'(0)} \tag{55}$$

[E^Q denotes expectation under Q], the strong law of large numbers and the inequality

$$T([nx], [n\varepsilon]) \geqslant \sum_{i=[nx]}^{0} p_i^{-1} v_{i,1}$$

imply that $\gamma(x, \varepsilon) \geqslant 1$ for all $x \leqslant -a'(0)$ and $\varepsilon > 0$. This gives the first part of Eq. (10), namely that $g(x) = 0$ for $x \leqslant -a'(0)$.

Next we establish

Lemma 1. $g(0-) = c$.

Proof. For $k = -1, -2, -3,\dots$ let $M_k = \min\{p_k,\dots, p_0\}$ and let $i(k)$ be the least index in $\{k, k+1,\dots, 0\}$ such that $p_{i(k)} = M_k$. For any $x < 0$, if n is large enough so that $[nx] \leqslant k$,

$$T([nx], [nc]) \geqslant M_k^{-1} \sum_{j=1}^{[nc]} v_{i(k), j}$$

and consequently $\gamma(x, c) \geqslant cE[M_k^{-1}]$. Letting $k \to -\infty$ gives $\gamma(x, c) \geqslant 1$ for all $x < 0$, so that $g(x) \leqslant c$ for all $x < 0$. This gives $g(0-) \leqslant c$.

For the converse inequality, we use the limit of the corresponding interface problem without the disorder. Let $S(i, j)$ be the quantity in Eq. (51) with $p_m \equiv 1$, in other words, without disorder.

Its limit was calculated by Rost in 1981:[25]

$$\lim_{n \to \infty} n^{-1} S([nx], [ny]) = (\sqrt{|x|} + \sqrt{y})^2 \tag{56}$$

Since

$$T([nx],[ny]) \leqslant c^{-1} S([nx],[ny]) \tag{57}$$

we get $\gamma(x, y) \leqslant c^{-1}(\sqrt{|x|}+\sqrt{y})^2$. Let $y<c$, and pick $x<0$ close enough to 0 so that $\sqrt{|x|}+\sqrt{y} \leqslant \sqrt{c}$. Then $g(x) \geqslant y$, and letting first $x \nearrow 0$ and then $y \nearrow c$ gives $g(0-) \geqslant c$.

It follows that g can be regarded as a continuous function on $(-\infty, 0]$, and because g is compactly supported, g is in fact uniformly continuous.

4.2. Construction and Coupling

The key to the proof of our theorems is a coupling of infinitely many copies of the interface process $Z(i, t)$ with an exclusion process. First we describe a rigorous construction of the exclusion process through a graphical representation.[4, 9, 17] The discussion in this section is formulated in terms of the TASEP $\sigma(i, t)$ instead of the interface process $h(i, t)$. Of course, due to Eq. (19), defining $\sigma(i, t)$ is the same as defining $h(i, t)$.

As explained before Theorem 2, we assume that in the background we have a probability space $(\Omega, \mathscr{F}, P)$ on which are defined the random rates $\mathbf{p}=\{p_i : i \in \mathbf{Z}\}$, a collection of point processes $\{\mathscr{D}_i : i \in \mathbf{Z}\}$, and the initial configuration $\sigma(\cdot, 0)=\{\sigma(i, 0) : i \in \mathbf{Z}\}$ of the exclusion process. The probability measure P satisfies these assumptions: (i) The marginal distribution of $\mathbf{p}$ is i.i.d. with common marginal $f(p)\, dp$. (ii) Given $\mathbf{p}$, the point processes $\{\mathscr{D}_i\}$ and the configuration $\sigma(\cdot, 0)$ are independent. (iii) Given $\mathbf{p}$, $\mathscr{D}_i$ is a Poisson point process on $[0, \infty)$ with rate p_i, and the processes $\{\mathscr{D}_i\}$ are mutually independent.

A comment: As emphasized in the paragraph before Assumption A in Sect. 2.1, the joint distribution of $\mathbf{p}$ and $\sigma(\cdot, 0)$ is arbitrary subject to the requirement that $\mathbf{p}$ have the correct i.i.d. $f(p)\, dp$ marginal. For Theorems 2 and 3 the probability space $(\Omega, \mathscr{F}, P)$ is further enlarged to support all the initial configurations $\sigma_n(\cdot, 0)$.

The Poisson process $\mathscr{D}_i$ represents the potential jump times of particle $\sigma(i)$. About the realizations $\{\mathscr{D}_i\}$ of the potential jump times we make these assumptions:

(a) Each $\mathscr{D}_i$ has only finitely many epochs in a bounded time interval.

(b) There are no simultaneous jump attempts.

(c) There are arbitrarily large i_0 and t_0 such that $\mathscr{D}_{-i_0}$ and $\mathscr{D}_{i_0}$ have no epochs in $[0, t_0]$. (58)

These assumptions are satisfied by almost every realization $\{\mathscr{D}_i\}$, by standard properties of Poisson point processes.

Now assume that realizations of $\mathbf{p}$, $\{\mathscr{D}_i\}$, and $\sigma(\cdot, 0)$ are chosen from the probability distribution P, and that assumptions (58) are in force. The process $\sigma(\cdot, t) = \{\sigma(i, t): i \in \mathbf{Z}\}$, $t \geqslant 0$, is then defined as a function of $\{\mathscr{D}_i\}$ and $\sigma(\cdot, 0)$, according to this rule: Suppose τ is an epoch of $\mathscr{D}_i$ and that $\sigma(i, \tau-) = \ell$. Then $\sigma(i, \tau) = \ell + 1$ (particle $\sigma(i)$ jumps from site ℓ to site $\ell + 1$ at time τ) if $\sigma(i+1, \tau-) \geqslant \ell + 2$ (site $\ell + 1$ is vacant). Otherwise the jump is suppressed, and $\sigma(i, \tau) = \ell$. In both cases particle $\sigma(i)$ then stays put until the next epoch of $\mathscr{D}_i$, at which time it attempts a new jump to the right.

Assumptions (58) allow us to argue that this rule defines the evolution unambiguously for all times $0 \leqslant t < \infty$ and all i: Given i and t, part (c) of (58) gives i_0 and t_0 so that $-i_0 < i < i_0$ and $t < t_0$, and so that $\mathscr{D}_{-i_0}$ and $\mathscr{D}_{i_0}$ have no epochs in $[0, t_0]$. Consequently $\sigma(-i_0, t) = \sigma(-i_0, 0)$ and $\sigma(i_0, t) = \sigma(i_0, 0)$ for all $t \in [0, t_0]$ and the evolution of particles $\sigma(j)$ for $-i_0 < j < i_0$ is isolated from the rest of the process up to time t_0. By parts (a) and (b) of (58), the locations $\sigma(j, t)$ for $-i_0 < j < i_0$ and $t \in [0, t_0]$ can be computed by applying the rule in temporal order to the finitely many potential jump times in $\bigcup_{-i_0 < j < i_0} \mathscr{D}_j \cap [0, t_0]$. In particular, the motion of particle $\sigma(i)$ is then defined up to time t.

A useful property of this construction is that it preserves orderings between two processes:

Lemma 2. Suppose the probability space $(\Omega, \mathscr{F}, P)$ contains two initial configurations $\sigma'(\cdot, 0)$ and $\sigma''(\cdot, 0)$ that satisfy $\sigma'(i, 0) \leqslant \sigma''(i, 0)$ for all $i \in \mathbf{Z}$ with probability 1. Then if the processes $\sigma'(\cdot, t)$ and $\sigma''(\cdot, t)$ are constructed according to the description above, we have $\sigma'(i, t) \leqslant \sigma''(i, t)$ for all $i \in \mathbf{Z}$ and $t \geqslant 0$, with probability 1.

Proof. Choose again i_0 and t_0 as in part (c) of assumption (58). We shall prove that the ordering $\sigma'(i, t) \leqslant \sigma''(i, t)$ holds for $-i_0 \leqslant i \leqslant i_0$ and $t \leqslant t_0$. It holds for $i = \pm i_0$ because particles $\sigma'(\pm i_0)$ and $\sigma''(\pm i_0)$ do not attempt to jump during the time interval $[0, t_0]$. By parts (a) and (b) of Eq. (58), we may do induction over the jump times. Suppose $\tau \in \mathscr{D}_i$ is the first epoch at which the ordering is violated, for $-i_0 \leqslant i \leqslant i_0$ and $\tau \leqslant t_0$. Then we must have $\sigma'(i, \tau) > \sigma''(i, \tau)$ but $\sigma'(i, \tau-) = \sigma''(i, \tau-)$. Particle $\sigma'(i)$ jumped at time τ, but the jump of particle $\sigma''(i)$ was blocked by $\sigma''(i+1)$. This implies that $\sigma'(i+1, \tau-) > \sigma''(i+1, \tau-)$. Since τ is by assumption the first time the ordering is violated for $-i_0 \leqslant i \leqslant i_0$, we conclude that $i+1 > i_0 \geqslant i$, which implies that $i = i_0$. But there are no epochs in $\mathscr{D}_{i_0}$ up to time t_0, and we have a contradiction. Thus there can be no first violation of the ordering for $-i_0 \leqslant i \leqslant i_0$ and $t \leqslant t_0$. Since i_0 and t_0 can be taken arbitrarily large, the proof is complete.

Consider a fixed initial configuration $\{\sigma(j, 0): j \in \mathbf{Z}\}$. For each initial location $\sigma(j, 0)$ construct an auxiliary exclusion process denoted by ζ^j. The initial configuration of the process ζ^j is centered at $\sigma(j, 0)$ in the following sense: Initially all sites in $(-\infty, \sigma(j, 0)]$ are occupied, and all sites in $(\sigma(j, 0), \infty)$ are empty. The particles of the ζ^j-process are labeled by $i = 0, -1, -2, -3,...,$ so the initial configuration of ζ^j is given by

$$\zeta^j(i, 0) = \sigma(j, 0) + i \qquad \text{for} \quad i \leqslant 0 \tag{59}$$

The particles $\zeta^j(i)$ for $i > 0$ are not needed; alternatively, we may think that they reside at ∞. The dynamical rule for ζ^j is this:

$$\text{particle } \zeta^j(i) \text{ reads its jump commands from the Poisson process } \mathscr{D}_{i+j} \tag{60}$$

In particular, particle $\zeta^j(i)$ jumps at rate p_{i+j}. Subject to these rules, the processes ζ^j, $j \in \mathbf{Z}$, are constructed as was described above. The processes σ and $\{\zeta^j\}$ are all defined on the same probability space $(\Omega, \mathscr{F}, P)$, and they use the same Poisson jump times. Notice though that the processes are invisible to each other, i.e., particles of one process do not interfere with the jumps of the other processes.

The key property of this coupling is here:

Lemma 3. The equality

$$\sigma(k, t) = \inf_{j:\, j \geqslant k} \zeta^j(k - j, t) \tag{61}$$

holds for all $k \in \mathbf{Z}$ and $t \geqslant 0$, almost surely.

Proof. The exclusion rule (18) and (59) imply that (61) holds at time 0. The point of (60) is that for each $j \geqslant k$, particles $\zeta^j(k-j)$ and $\sigma(k)$ read their jump commands from the same Poisson process $\mathscr{D}_k$. As in the previous proofs, assumptions (58) reduce this proof to showing inductively that there cannot be a jump that violates (61). See refs. 27 or 29 for more details of this kind of an argument.

Let $i < j$, and set

$$\sigma'(\ell, t) = \begin{cases} \zeta^j(\ell + i - j, t) + \sigma(i, 0) - \sigma(j, 0) + j - i, & \ell \leqslant j - i \\ \infty, & \ell > j - i \end{cases} \tag{62}$$

and

$$\sigma''(\ell, t) = \begin{cases} \zeta^i(\ell, t), & \ell \leqslant 0 \\ \infty, & \ell > 0 \end{cases} \tag{63}$$

By Eq. (60) both $\sigma'(\ell)$ and $\sigma''(\ell)$ read their jump commands from $\mathscr{D}_{\ell+i}$, and by Eq. (59) $\sigma'(\ell, 0) \leqslant \sigma''(\ell, 0)$ for all $\ell \in \mathbf{Z}$. An application of Lemma 2 gives

$$\zeta^i(k-i, t) \geqslant \zeta^j(k-j, t) + \sigma(i, 0) - \sigma(j, 0) + j - i \tag{64}$$

for all $j > i \geqslant k$ and $t \geqslant 0$. This inequality will be used later.

Finally we make contact with the interface process $Z(i, t)$ of Sect. 4.1. For $j \in \mathbf{Z}$, $i \leqslant 0$ and $t \geqslant 0$ define

$$\begin{aligned} Z^j(i, t) &= \text{the displacement of particle } \zeta^j(i) \text{ by time } t \\ &= \zeta^j(i, t) - \zeta^j(i, 0) \\ &= \zeta^j(i, t) - \sigma(j, 0) - i \end{aligned} \tag{65}$$

This definition removes the effect of the initial location $\sigma(j, 0)$ on the distribution of the process ζ^j.

In the next lemma the initial distribution of the process σ is arbitrary. We introduce notation for a translation of $\mathbf{p}$: $(\theta^j \mathbf{p})_i = p_{i+j}$. Recall the setting of Sect. 4.1: The process $Z(i, t)$ is defined by (52) in the last-passage picture, and the probability distribution in the background is Q.

Lemma 4. Under $P(\cdot \mid \mathbf{p})$, $Z^j(i, t)$ is independent of $\sigma(j, 0)$, and the distribution of $Z^j(i, t)$ equals the distribution of $Z(i, t)$ under $Q(\cdot \mid \theta^j \mathbf{p})$. In particular, $Z^j(i, t)$ under P and $Z(i, t)$ under Q are equal in distribution.

Proof. Fix $j \in \mathbf{Z}$. For $i \leqslant 0$ and $k \geqslant 1$ set

$$T^j(i, k) = \inf\{t > 0 : Z^j(i, t) \geqslant k\}, \qquad k \geqslant 1$$

the time when $Z^j(i, t)$ jumps from $k-1$ to k. By Eq. (65), this is the time when particle $\zeta^j(i)$ executes its kth jump. Then $\max\{T^j(i+1, k), T^j(i, k-1)\}$ is the time when $\zeta^j(i+1)$ has executed its kth jump and $\zeta^j(i)$ has executed its $(k-1)$st jump. At this moment the next site ahead of $\zeta^j(i)$ is vacant and nothing obstructs the kth jump of $\zeta^j(i)$. Let $\tau^j_{i,k}$ denote the amount of time that $\zeta^j(i)$ waits after this moment to execute its kth jump. These considerations lead to the equation

$$T^j(i,k) = \tau^j_{i,k} + \max\{T^j(i+1,k), T^j(i,k-1)\} \tag{66}$$

for $i \leqslant 0, k \geqslant 1$, together with the boundary values $T^j(i,0) = T^j(1,k) = 0$.

According to the jump rule (60) and the construction, under the quenched probability measure $P(\cdot \mid \mathbf{p})$ each $\tau^j_{i,k}$ is exponentially distributed with rate p_{i+j} and independent of the other waiting times for this fixed j. Thus the process $\{\tau^j_{i,k}: i \leqslant 0, k \geqslant 1\}$ under $P(\cdot \mid \mathbf{p})$ has exactly the same distribution as the process $\{p_i^{-1} v_{i,k}: i \leqslant 0, k \geqslant 1\}$ under $Q(\cdot \mid \theta^j \mathbf{p})$.

The time when $Z(i,t)$ jumps from $k-1$ to k is the last-passage time $T(i,k)$ defined by Eq. (51). This definition can be equivalently formulated like this: Set $T(i,0) = T(1,k) = 0$, and then inductively

$$T(i,k) = p_i^{-1} v_{i,k} + \max\{T(i+1,k), T(i,k-1)\} \tag{67}$$

for $i \leqslant 0, k \geqslant 1$. Comparison of Eq. (66) and (67) shows that the process $\{T^j(i,k): i \leqslant 0, k \geqslant 1\}$ under $P(\cdot \mid \mathbf{p})$ and the process $\{T(i,k): i \leqslant 0, k \geqslant 1\}$ under $Q(\cdot \mid \theta^j \mathbf{p})$ are equal in distribution. This implies the equality in distribution claimed in the lemma since $Z^j(i,t)$ and $Z(i,t)$ are functions of these processes [see Eq. (52)].

This description of the distribution of $Z^j(i,t)$ under $P(\cdot \mid \mathbf{p})$ is not changed by conditioning on $\sigma(j,0)$, so the independence of $Z^j(i,t)$ and $\sigma(j,0)$ holds.

To go from quenched distributions to annealed distributions we average over the disorder $\mathbf{p}$. This averaging removes the effect of the translation $\theta^j \mathbf{p}$, and we get the last statement of the lemma.

According to the last statement of the lemma, under annealed probabilities $Z^j(i,t)$ has the same distribution as $Z(i,t)$ for all j. In particular, they all have the same macroscopic limit $g(x)$ in annealed probability, and can be used to calculate g. This will be done in the next section. Let us rewrite Eq. (61) in the form

$$\sigma(k,t) = \inf_{j:\, j \geqslant k} \{\sigma(j,0) + k - j + Z^j(k-j,t)\} \tag{68}$$

obtained by combining Eq. (61) with Eq. (65).

4.3. Completion of the Proof of Theorem 1

To identify the function g we apply the coupling (68) to the special case where the process σ is in equilibrium. As above, Q is the probability distribution of the last-passage construction of Sect. 4.1, while P is the probability measure under which the processes σ, ζ^j, and Z^j are constructed in Sect. 4.2. Before we start, a technical lemma that allows us to reduce the set of indices in taking the infimum in Eq. (68).

Lemma 5. Let $t>0$ and $r>a'(0)\,t$. Then, for some constant $C=C(t,r)>0$,

$$Q\{Z(-[nr], nt)>0\} \leqslant e^{-Cn}$$

for all n.

Proof. $Z(-[nr], nt)>0$ if and only if

$$\sum_{i=-[nr]}^{0} p_i^{-1} v_{i,1} \leqslant nt \tag{69}$$

This sum has expectation roughly $nr/a'(0)$ [recall Eq. (55)], which exceeds nt by an amount of order n by the choice of r. By standard large deviation arguments the event in Eq. (69) has probability at most e^{-Cn} for some constant $C>0$. C depends on r, t, and the distribution $f(p)$.

Pick and fix $u<u^*$, and let $a(u)$ be the corresponding velocity. The random initial configuration $\{\sigma(j,0)\}$ has a joint distribution with the disorder, defined as follows: $\sigma(0,0)=0$ with probability 1. Conditional on $\{p_j\}$, the gap sizes $\eta(i,0)=\sigma(i+1,0)-\sigma(i,0)-1$ are independent with distribution (20), with $v=a(u)$, so that $u=\overline{\langle\eta_i\rangle}$. Then almost surely

$$\lim_{n\to\infty} \{n^{-1}\sigma([nx],0)-n^{-1}[nx]\} = xu \tag{70}$$

for all $x\in\mathbf{R}$. In this steady state particle $\sigma(0)$ jumps according to a Poisson$(a(u))$ process, so almost surely

$$\lim_{n\to\infty} n^{-1}\sigma(0,n)=a(u) \tag{71}$$

On the other hand, by Eq. (68),

$$\sigma(0,n)=\inf_{j:\, j\geqslant 0} \{\sigma(j,0)-j+Z^j(-j,n)\} \tag{72}$$

Here is how Lemma 5 helps us:

Lemma 6. Let $r > a'(0)$. Then P-almost surely

$$\sigma(0, n) = \min_{j:\, 0 \leqslant j \leqslant [rn]} \{\sigma(j, 0) - j + Z^j(-j, n)\} \tag{73}$$

for large enough n.

Proof. By Lemma 5 and the Borel–Cantelli lemma, $Z^{[nr]}(-[nr], n) = 0$ for large enough n. [Recall the equality under annealed distributions given in Lemma 4.] For $j > [nr]$ the exclusion rule and the nonnegativity of $Z^j(-j, n)$ imply

$$\begin{aligned}\sigma(j, 0) - j + Z^j(-j, n) &\geqslant \sigma([nr], 0) - [nr] \\ &= \sigma([nr], 0) - [nr] + Z^{[nr]}(-[nr], n)\end{aligned}$$

Consequently the indices $j > [nr]$ cannot contribute to the infimum in Eq. (72).

Pick a partition $0 = s_0 < s_1 < \cdots < s_m = r$ such that $s_{i+1} - s_i < \delta$ for each i. Rewrite Eq. (73) as

$$\sigma(0, n) = \min_{0 \leqslant i < m} \min_{[ns_i] \leqslant j \leqslant [ns_{i+1}]} \zeta^j(-j, n)$$

Equation (64), and the inequality $\sigma(j, 0) - j \geqslant \sigma([ns_i], 0) - [ns_i]$ for $[ns_i] \leqslant j \leqslant [ns_{i+1}]$, give

$$\begin{aligned}\sigma(0, n) \geqslant \min_{0 \leqslant i < m} \{&\zeta^{[ns_{i+1}]}(-[ns_{i+1}], n) + [ns_{i+1}] - [ns_i] \\ &+ \sigma([ns_i], 0) - \sigma([ns_{i+1}], 0)\}\end{aligned} \tag{74}$$

Equation (65) and multiplying through by $1/n$ give

$$\frac{1}{n}\sigma(0, n) \geqslant \min_{0 \leqslant i < m} \left\{\frac{1}{n}\sigma([ns_i], 0) - \frac{1}{n}[ns_i] + \frac{1}{n} Z^{[ns_{i+1}]}(-[ns_{i+1}], n)\right\}$$

Let $n \to \infty$. Each term above converges in annealed probability, and a minimum over finitely many random variables is preserved by the limit. Apply Eqs. (70), (71), and (7) to arrive at

$$\begin{aligned}a(u) &\geqslant \min_{0 \leqslant i < m} \{us_i + g(-s_{i+1})\} \\ &\geqslant \inf_{x > 0} \{ux + g(-x)\} - u\delta\end{aligned}$$

Letting $\delta \searrow 0$ finally gives

$$a(u) \geqslant \inf_{x>0} \{ux + g(-x)\}$$

This was the difficult half. Equation (72) immediately gives

$$\frac{1}{n}\sigma(0, n) \leqslant \frac{1}{n}\sigma([nx], 0) - \frac{1}{n}[nx] + \frac{1}{n} Z^{[nx]}(-[nx], n)$$

for all $x > 0$. Passing to the limit yields

$$a(u) \leqslant \inf_{x>0} \{ux + g(-x)\}$$

We have verified that

$$a(u) = \inf_{x>0} \{ux + g(-x)\}, \qquad 0 \leqslant u < u^* \tag{75}$$

To prove Theorem 1 it remains to deduce Eq. (8) from Eq. (75). First note that Eq. (75) holds also for $u \geqslant u^*$: This is because the right-hand side of Eq. (75) defines a nondecreasing function of u with upper bound c [let $x \searrow 0$ in Eq. (75) and use Lemma 1]. Since $a(u^* -) = c$ by the definition (3) of u^*, Eq. (75) must give $a(u) = c$ for $u \geqslant u^*$, and thereby agree with the original definition of $a(u)$. Secondly, let Eq. (75) define $a(u)$ also for $u < 0$, giving $a(u) = -\infty$. The function $-a(u)$ thus defined is a lower semicontinuous convex function on all of **R**.

Next, extend g to a lower semicontinuous convex function on all of **R** by defining $g(0) = c$, $g(x) = \infty$ for $x > 0$. This does not affect Eq. (75), which now says that $-a(u)$ is the convex dual of $g(x)$. By the standard duality theory of convex analysis[24]

$$g(x) = \sup_u \{xu + a(u)\}$$

which gives Eq. (8). This completes the proof of Theorem 1.

The remaining properties of g and the classification into Cases I–III follow from convex-analytic considerations. By Eq. (11) $a(u)$ has a continuous derivative for $0 < u < u^*$. It follows that $g(x)$ is strictly convex for $-a'(0) < x < x^*$. From this follows the remaining part of (10): $g(x) > 0$ for $-a'(0) < x < 0$. For $x < x^*$,

$$g(x) = xu + a(u)$$

if and only if $a'(u) = -x$ or equivalently $g'(x) = u$. In the interesting case $x^* < 0$, $g(x^*) = x^*u^* + c$ and $g'(x^*-) = u^*$, so $g'(x)$ must equal u^* for $x^* < x < 0$. This is because the slope of g is nondecreasing by convexity, and yet the graph of g must connect the point $(x^*, x^*u^* + c)$ to the point $(0, c)$. The linear segment in g with slope u^* over the internal $(x^*, 0)$ corresponds to a jump in $a'(u)$ from $-x^*$ to 0 across $u = u^*$.

4.4. An Annealed Large Deviation Estimate

We return to the development that leads to Theorem 2. This section is devoted to the proof of the following proposition:

Proposition 1. Let $t > 0$, $x < 0$, and $\varepsilon > 0$. Then there is a constant C such that

$$Q\{|Z([nx], nt) - ntg(x/t)| \geqslant n\varepsilon\} \leqslant e^{-Cn} \tag{76}$$

for all large enough n.

Proof. We begin the proof with

$$Q\{Z([nx], nt) \geqslant [nr]\} \leqslant e^{-Cn} \tag{77}$$

for some $C > 0$, whenever $r > tg(x/t)$. Its proof uses the last-passage picture. $r > tg(x/t)$ is equivalent to $t < \gamma(x, r)$, and we shall prove that

$$Q\{T([nx], [nr]) \leqslant nt\} \leqslant e^{-Cn}$$

which is equivalent to Eq. (77). Since

$$\ell^{-1} E^Q[T([\ell x], [\ell r])] \to \gamma(x, r) \qquad \text{as} \quad \ell \to \infty$$

we can pick and fix an integer ℓ and a number $\delta > 0$ such that

$$E^Q[T([\ell x], [\ell r])] > \ell(t + \delta)$$

Let $\Lambda = \{[\ell x],..., 0\} \times \{1,..., [\ell r]\}$, and consider the translates

$$\Lambda^{(j)} = (j[\ell x] - j, j[\ell r]) + \Lambda, \qquad j = 0, 1, 2,...$$

Each rectangle $\Lambda^{(j)}$ has a last-passage time $\tau^{(j)}$, distributed exactly as $T([\ell x], [\ell r])$ under Q, and defined as in Eq. (51) but with paths π that connect the lower right and upper left corners of $\Lambda^{(j)}$.

Let n be large, and let k be the largest positive integer that satisfies both $[nx] \leqslant k([\ell x]-1)$ and $[nr] \geqslant k[\ell r]$. If ℓ is chosen large enough relative to $|x|$ and r, such a k will satisfy $k \geqslant n/\ell - 2$ for all large n. It then follows that for large enough n

$$nt \leqslant (1-\delta_1)\, kE^Q[\tau^{(0)}]$$

for some $\delta_1 > 0$. Now $T([nx], [nr]) \geqslant \sum_0^{k-1} \tau^{(j)}$. Since the $\{\tau^{(j)}\}$ are i.i.d. under the annealed probability distribution Q, the required estimate follows from a standard large deviation bound for nonnegative i.i.d. random variables:

$$\begin{aligned} &Q\{T([nx], [nr])) \leqslant nt\} \\ &\quad \leqslant Q\left\{\sum_{j=0}^{k-1} \tau^{(j)} \leqslant (1-\delta_1)\, kE^Q[\tau^{(0)}]\right\} \\ &\quad \leqslant \exp(-C'k) \\ &\quad \leqslant \exp(-Cn) \end{aligned}$$

This completes the proof of Eq. (77).

It remains to prove the lower tail estimate,

$$Q\{Z([nx], nt) < nr\} \leqslant e^{-Cn} \tag{78}$$

for some $C > 0$, whenever $r < tg(x/t)$. This utilizes the coupling with TASEP, and requires some preliminary work.

For $\beta \leqslant 0$ and $u \geqslant 0$ let

$$G_{\beta, u}(p) = \log \frac{p - a(u)}{p - e^{\beta} a(u)}, \qquad p \in [c, 1] \tag{79}$$

$G_{\beta, u}$ is a nonpositive function. Let

$$\gamma(\beta, u) = E^Q[G_{\beta, u}(p_0)] = \int_c^1 G_{\beta, u}(p)\, f(p)\, dp \tag{80}$$

denote the expectation of $G_{\beta, u}$ under the disorder distribution. One can check that $\gamma(\beta, u) > -\infty$ for all $\beta \leqslant 0$ and $u \in [0, u^*)$, and also at $u = u^*$ in case $u^* < \infty$. The relevance of $G_{\beta, u}$ is that if the TASEP configuration $\{\sigma(j, 0)\}$ has the independent gap distributions (20) with $v = a(u)$, then for $j > 0$ and $\beta \leqslant 0$ we have the quenched moment generating function

$$E^P[\exp(\beta\sigma(j, 0) - \beta j) \mid \mathbf{p}] = \exp\left(\sum_{i=0}^{j-1} G_{\beta, u}(p_i)\right) \tag{81}$$

Lemma 7. Given $x<0, t>0$, and $r<tg(x/t)$, there exist $\beta<0$, $u \in [0, u^*)$, and $\delta_0>0$ such that

$$-x\gamma(\beta, u) > ta(u)(e^{\beta}-1) - \beta r + 2\delta_0$$

Proof. By the assumption $r<tg(x/t)$ and by the duality formula (8) we may pick $u \in [0, u^*)$ so that $ux + ta(u) > r$. [By continuity formula (8) remains valid even if u is restricted to the interval $[0, u^*)$.] Let

$$\phi_1(\beta) = -x\gamma(\beta, u)$$

and

$$\phi_2(\beta) = ta(u)(e^{\beta}-1) - \beta r$$

Since $\phi_1(0) = \phi_2(0) = 0$, it suffices to show that $\phi_2'(0) > \phi_1'(0)$. We get

$$\phi_1'(0) = -xa(u) \int_c^1 \frac{f(p)}{p-a(u)}\, dp = -xu$$

by (4) or (21), while

$$\phi_2'(0) = ta(u) - r > -xu$$

by the choice of u.

We return to the proof of the lower tail estimate (78). Let $\beta<0$, $u \in [0, u^*)$, and $\delta_0>0$ be the numbers given by Lemma 7, and let $G(p) = G_{\beta, u}(p)$. Let $A_n = \{Z([nx], nt) < nr\}$ denote the event in question. By an application of Chebyshev's inequality,

$$Q(A_n \mid \mathbf{p}) \leqslant e^{-\beta rn} E^Q[\exp\{\beta Z([nx], nt)\} \mid \mathbf{p}]$$

Thus we start by conditioning on $\mathbf{p}$:

$$\begin{aligned} Q(A_n) &= \int Q(A_n \mid \mathbf{p})\, Q(d\mathbf{p}) \\ &\leqslant e^{-n\delta_0} + Q\{\mathbf{p}: Q(A_n \mid \mathbf{p}) \geqslant e^{-n\delta_0}\} \\ &\leqslant e^{-n\delta_0} + Q\{\mathbf{p}: E^Q[\exp\{\beta Z([nx], nt)\} \mid \mathbf{p}] \geqslant e^{-n(\delta_0 - \beta r)}\} \end{aligned}$$

It suffices now to show that, for some constant C_1,

$$Q\{\mathbf{p}: E^Q[\exp\{\beta Z([nx], nt)\} \mid \mathbf{p}] \geqslant e^{-n(\delta_0 - \beta r)}\} \leqslant e^{-C_1 n}$$

We may translate $\mathbf{p}$ by $\theta^{-[nx]}$ inside the probability because the distribution of $\mathbf{p}$ under Q is i.i.d. Let $\sigma(i, t)$ denote an exclusion process whose distribution is the steady state (20) with $v = a(u)$, so that the expected gap equals u. Then use the equality in distribution given in Lemma 4, and the above probability turns into

$$Q\{\mathbf{p}\colon E^P[\exp\{\beta Z^{-[nx]}([nx], nt)\} \mid \mathbf{p}] \geqslant e^{-n(\delta_0 - \beta r)}\} \tag{82}$$

Next, use the independence of $Z^{-[nx]}([nx], nt)$ and $\sigma(-[nx], 0)$ under $P(\cdot \mid \mathbf{p})$ and formula (81) to turn (82) into

$$Q\left\{\mathbf{p}\colon E^P[\exp\{\beta\sigma(-[nx], 0) + \beta[nx] + \beta Z^{-[nx]}([nx], nt)\} \mid \mathbf{p}] \right.$$
$$\left. \geqslant \exp\left(-n(\delta_0 - \beta r) + \sum_{i=0}^{-[nx]-1} G(p_i)\right)\right\} \tag{83}$$

By the coupling (68)

$$\sigma(0, nt) \leqslant \sigma(-[nx], 0) + [nx] + Z^{-[nx]}([nx], nt)$$

so the probability in Eq. (83) is bounded above by

$$Q\left\{\mathbf{p}\colon E^P[\exp\{\beta\sigma(0, nt)\} \mid \mathbf{p}] \right.$$
$$\left. \geqslant \exp\left(-n(\delta_0 - \beta r) + \sum_{i=0}^{-[nx]-1} G(p_i)\right)\right\} \tag{84}$$

Under $P(\cdot \mid \mathbf{p})$ $\sigma(0, nt)$ has a Poisson$(a(u)nt)$ distribution, and consequently

$$E^P[\exp\{\beta\sigma(0, nt)\} \mid \mathbf{p}] = \exp\{na(u)\, t(e^\beta - 1)\}$$

With this the probability in Eq. (84) simplifies to

$$Q\left\{\mathbf{p}\colon \sum_{i=0}^{-[nx]-1} G(p_i) \leqslant n[a(u)\, t(e^\beta - 1) + \delta_0 - \beta r]\right\} \tag{85}$$

By Eq. (80) and Lemma 7 there is a $\delta_1 > 0$ such that the probability in Eq. (85) is bounded above by

$$Q\left\{\mathbf{p}\colon \sum_{i=0}^{-[nx]-1} G(p_i) \leqslant -[nx](E^Q[G(p_0)] - \delta_1)\right\} \tag{86}$$

(Recall that $x<0$.) It remains to argue that this probability is exponentially small in n. We have

$$0 \geqslant G(p_i) \geqslant \log(c - a(u))$$

Since $u<u^*$, $a(u)<c$, and we see that the $G(p_i)$'s are bounded i.i.d. random variables. By standard large deviation arguments, the probability in Eq. (86) is bounded by $e^{-C_1 n}$.

This completes the proof of Eq. (78) and also the proof of Proposition 1.

4.5. Proof of Theorem 2

As indicated in the paragraph following the statement of Theorem 2, the proof is formulated for the particular construction of the interface process in terms of the TASEP. Assume that the disorder $\mathbf{p}$, the Poisson point processes $\{\mathscr{D}_i\}$, and the initial interfaces $\{h_n(\cdot, 0): n \in \mathbf{N}\}$ are defined on a probability space $(\Omega, \mathscr{F}, P)$. The initial exclusion configurations are defined by

$$\sigma_n(i, 0) = h_n(i, 0) + i \tag{87}$$

The exclusion processes $\sigma_n(i, t)$ are constructed as described in Sect. 4.2, so that for each n particle $\sigma_n(i)$ reads its jump commands from Poisson process $\mathscr{D}_i$. The interface processes $h_n(i, t)$ are defined in terms of the TASEP's by $h_n(i, t) = \sigma_n(i, t) - i$.

As in Sect. 4.2, for each n we have the auxiliary TASEP's ζ_n^j that satisfy

$$\zeta_n^j(i, 0) = \sigma_n(j, 0) + i \quad \text{for } i \leqslant 0 \text{ at time zero, and} \tag{88}$$

$$\text{particle } \zeta_n^j(i) \text{ reads its jump commands from } \mathscr{D}_{i+j} \tag{89}$$

In the coupling we use the interface processes Z_n^j defined by

$$Z_n^j(i, t) = \zeta_n^j(i, t) - \zeta_n^j(i, 0) = \zeta_n^j(i, t) - \sigma_n(j, 0) - i \tag{90}$$

We begin by defining the event of full measure on which the convergence will be proved to take place. Pick and fix positive numbers $\delta_m > 0$ such that $\delta_m \searrow 0$ as $m \nearrow \infty$. Define a sequence of partitions of the real line by $y_\ell^m = \ell \delta_m$, $\ell \in \mathbf{Z}$.

Lemma 8. There is an event $\Theta_0 \subseteq \Omega$ of full P-probability such that, on this event, the coupling equality (61) and the ordering (64) hold always, and the following statements hold for all $t>0$, all $m>0$, and all $k<\ell$:

$$\begin{aligned} &Z_n^{[ny_\ell^m]}([ny_k^m]-[ny_\ell^m], nt) \\ &\quad = 0 \qquad \text{for large enough } n, \text{ if } y_\ell^m - y_k^m > a'(0)\, t \end{aligned} \tag{91}$$

and

$$\lim_{n\to\infty} \frac{1}{n} Z_n^{[ny_\ell^m]}([ny_k^m]-[ny_\ell^m], nt) = tg\left(\frac{y_k^m - y_\ell^m}{t}\right) \tag{92}$$

Proof. Assuming Eq. (61) and (64) simultaneously for all processes requires that we discard those realizations of $\{\mathscr{D}_i\}$ that do not satisfy Eq. (58), and this is only a zero-measure event. The exponential estimates of Lemma 5 and Proposition 1 are valid here because by Lemma 4, $Z_n^{[ny_\ell^m]}([ny_k^m]-[ny_\ell^m], nt)$ under P and $Z([ny_k^m]-[ny_\ell^m], nt)$ under Q are equal in distribution. Use the Borel–Cantelli lemma to define Θ_0 by requiring statements (91)–(92) simultaneously for all rational $t>0$ and all integers k, ℓ, and m. The extension to all $t>0$ follows from the monotonicity of the random variables in t and from the continuity of the function g.

Notice that the truth of this lemma does not depend on Assumptions A and B, even though the initial interface $h_n(\cdot, 0)$ enters into the definition of $Z_n^j(i, t)$ through Eq. (87) and (90). This is because the proof only uses the exponential estimates available for the deviations of $Z_n^j(i, t)$, and the marginal annealed distribution of $Z_n^j(i, t)$ is not affected by the index (j, n) or the initial interface $h_n(\cdot, 0)$.

The remainder of the proof has two separate parts. We first prove almost sure convergence under Assumption A. Let Θ be the subset of Θ_0 on which the convergence of Assumption A holds. By assumption, Θ is again of full P-measure. Pick and fix arbitrary $x \in \mathbf{R}$ and $t>0$. Our task is to prove that

$$\lim_{n\to\infty} n^{-1}\sigma_n([nx], nt) = U(x, t) + x \tag{93}$$

pointwise on the event Θ, when we define $U(x, t)$ by formula (15). Once this is done, the set Π_1 whose existence is asserted in the theorem is defined by

$$\Pi_1 = \{\mathbf{p}\colon P(\Theta \mid \mathbf{p}) = 1\} \tag{94}$$

Π_1 is of full P-measure because

$$\int P(\Theta \mid \mathbf{p})\, P(d\mathbf{p}) = P(\Theta) = 1$$

Let $b = b(m)$ be the index defined by $y_b^m \leqslant x < y_{b+1}^m$. For any $\ell \geqslant b+2$, Eq. (68) and the nondecreasingness of $Z^j(i, t)$ in the i-variable imply that

$$\sigma_n([nx], nt) \leqslant \sigma_n([ny_\ell^m], 0) + [nx] - [ny_\ell^m] + Z_n^{[ny_\ell^m]}([ny_{b+1}^m] - [ny_\ell^m], nt)$$

We required $\ell \geqslant b+2$ instead of $\ell \geqslant b+1$ so that the argument $[ny_{b+1}^m] - [ny_\ell^m]$ of $Z_n^{[ny_\ell^m]}$ will be strictly negative for large n. From this follows by Assumption (13), by Eq. (87), and by Eq. (92),

$$\begin{aligned} &\limsup_{n \to \infty} n^{-1} \sigma_n([nx], nt) \\ &\qquad \leqslant U_0(y_\ell^m) + tg((y_{b+1}^m - y_\ell^m)/t) + x \\ &\qquad \leqslant U_0(y_\ell^m) + tg((x - y_\ell^m)/t) + x + \varepsilon_m \end{aligned}$$

where $\varepsilon_m \to 0$ as $m \to \infty$, by the uniform continuity of g. By considering y_ℓ^m for all m and $\ell \geqslant b(m) + 2$ we get a dense subset of (x, ∞), so by the uniform continuity of g and by the right-continuity of U_0, we can conclude that

$$\limsup_{n \to \infty} n^{-1} \sigma_n([nx], nt) \leqslant U(x, t) + x \tag{95}$$

on the event Θ.

Now the other direction. Pick another index $d = d(m)$ such that $y_d^m - y_{b+1}^m > a'(0)\, t$. Then

$$Z_n^{[ny_d^m]}([ny_{b+1}^m] - [ny_d^m], nt) = 0$$

for large enough n by Eq. (91). Since $[nx] \leqslant [ny_{b+1}^m]$, also

$$Z_n^{[ny_d^m]}([nx] - [ny_d^m], nt) = 0 \tag{96}$$

for large enough n, by the monotonicity of $Z_n^j(i, nt)$ in the i-variable. By the argument of Lemma 6 we can discard $j > [ny_d^m]$ from the variational formula (68) and write, for large enough n,

$$\sigma_n([nx], nt) = \inf_{j:\, [nx] \leqslant j \leqslant [ny_d^m]} \{\sigma_n(j, 0) + [nx] - j + Z_n^j([nx] - j, nt)\}$$

Repeating the argument for Eq. (74) that followed Lemma 6 gives, with the partition $x < y^m_{b+1} < \cdots < y^m_d$,

$$\sigma_n([nx], nt) \geqslant \min_{b \leqslant \ell < d} \{\sigma_n([ny^m_\ell] \vee [nx], 0) + [nx] - ([ny^m_\ell] \vee [nx]) + Z^{[ny^m_{\ell+1}]}([nx] - [ny^m_{\ell+1}], nt)\}$$

Finally, again by the monotonicity of $Z^j_n(i, nt)$ in the i-variable,

$$\sigma_n([nx], nt) \geqslant \min_{b \leqslant \ell < d} \{\sigma_n([ny^m_\ell] \vee [nx], 0) + [nx] - ([ny^m_\ell] \vee [nx]) + Z^{[ny^m_{\ell+1}]}([ny^m_b] - [ny^m_{\ell+1}], nt)\}$$

We must use $\sigma_n([ny^m_\ell] \vee [nx], 0)$ instead of $\sigma_n([ny^m_\ell], 0)$ because otherwise $y^m_b < x$ could put $\lim_{n\to\infty} n^{-1}\sigma_n([ny^m_b]), 0)$ to the left of a jump in U_0 at x, and we would not get the correct limit. By the definition of the event Θ,

$$\begin{aligned}\liminf_{n\to\infty} n^{-1}\sigma_n([nx], nt) &\geqslant \min_{b \leqslant \ell < d} \{U_0(y^m_\ell \vee x) + x + tg((y^m_b - y^m_{\ell+1})/t)\} \\ &\geqslant U(x, t) + x - \varepsilon_m \end{aligned} \tag{97}$$

where again $\varepsilon_m \to 0$ as $m \to \infty$. Combining (95) and (97) gives (93) on the event Θ.

Now we tun to Assumption B. The event Θ_0 is defined as above by Lemma 8. The assumption is now that there is a full-measure set of disorders $\mathbf{p}$ such that $n^{-1}h_n([nx], 0) \to U_0(x)$ in $P(\cdot \mid \mathbf{p})$-probability, for all x. The set Π_1 of $\mathbf{p}$ required for the theorem is the subset of these that further satisfy the requirement

$$P(\Theta_0 \mid \mathbf{p}) = 1 \tag{98}$$

The task is to show that, for such a $\mathbf{p}$ and any (x, t),

$$\lim_{n\to\infty} n^{-1}\sigma_n([nx], nt) = U(x, t) + x \tag{99}$$

in $P(\cdot \mid \mathbf{p})$-probability. The argument proceeds much as the previous one did, utilizing the partition points y^m_ℓ, but by substituting convergence in probability for pointwise convergence. We leave the details to the reader.

4.6. Proof of Theorem 3

From Theorem 2, or directly from Eq. (99), we deduce the following: Under assumption (22),

$$\lim_{n \to \infty} n^{-1} \sigma_n([nx], nt) = V(x, t) \tag{100}$$

for a function $V(x, t)$ defined by

$$V(x, t) = \inf_{y:\, y \geqslant x} \left\{ V_0(y) + t\tilde{g}\left(\frac{x-y}{t}\right) \right\} \tag{101}$$

where $\tilde{g}(x) = g(x) + x$. [The equality $V(x, t) = U(x, t) + x$ replaces g by $\tilde{g}$ when Eq. (15) is turned into Eq. (101).] Applying Eq. (100) to the integral $\mu_n(nt, \phi)$ of a compactly supported smooth test function ϕ gives

$$\begin{aligned}
\lim_{n \to \infty} \mu_n(nt, \phi) &= \lim_{n \to \infty} n^{-1} \sum_{i \in \mathbf{Z}} \phi(n^{-1}\sigma_n(i, t)) \\
&= \lim_{n \to \infty} \int_{\mathbf{R}} \phi(n^{-1}\sigma_n([nx], t))\, dx \\
&= \int_{\mathbf{R}} \phi(V(x, t))\, dx \\
&= \int_{\mathbf{R}} \phi(x)\, \rho(x, t)\, dx
\end{aligned}$$

The last equality above follows from a change of variable, after the appropriate definitions: First define $R(x, t)$ as the inverse of $V(x, t)$:

$$R(x, t) = \inf\{\xi\colon V(\xi, t) \geqslant x\} = \sup\{\xi\colon V(\xi, t) \leqslant x\} \tag{102}$$

Note that the exclusion rule (18) and limit (100) imply that

$$V(x, t) - V(y, t) \geqslant x - y \qquad \text{for all} \quad x > y \tag{103}$$

This justifies the equality of the two last expressions in Eq. (102), and shows that $R(x, t)$ is a well-defined continuous, nondecreasing function with slope at most 1. Its derivative $\rho(x, t) = (\partial/\partial x)\, R(x, t)$ exists almost everywhere.

To prove Theorem 3, it only remains to show that $\rho(x, t)$ is the entropy solution of Eq. (25), in other words, to deduce Eq. (26) from Eq. (101) and (102). For this we need a lemma that relates j^* and $\tilde{g}$.

Lemma 9. The function j^* satisfies $j^*(x)=0$ for $x \geqslant c$. Also, $\tilde{g}(j^*(x))=x$ for $x \leqslant c$, and $j^*(\tilde{g}(\xi))=\xi$ for $\xi \leqslant 0$.

Proof. The first statement follows from Eq. (27) and the fact that $j'(\rho) \leqslant c$ for all $0 \leqslant \rho \leqslant 1$. For the second statement, first perform the change of variable $u=1/\rho-1$ in Eq. (8) to obtain

$$\tilde{g}(\xi)=\sup_{0 \leqslant \rho \leqslant 1} \{\xi/\rho+v(\rho)\} \tag{104}$$

Equations (27) and (104) imply that $\tilde{g}(j^*(x)) \leqslant x$ and $j^*(\tilde{g}(\xi)) \geqslant \xi$. Equalities follow from the monotonicity and bijectivity of $\tilde{g}$ from $(-\infty, 0]$ onto $(-\infty, c]$.

Now fix (x, t) with $t>0$. Our final argument proves that

$$R(x, t)=\sup_{y \geqslant x-ct} \left\{R_0(y)+tj^*\left(\frac{x-y}{t}\right)\right\} \tag{105}$$

This is the same as Eq. (26) because, due to the nondecreasingness of R_0 and the first statement of Lemma 9, the values $y<x-ct$ cannot contribute to the supremum.

Pick an arbitrary $y>x-ct$, and let $\eta=R_0(y)$. Since $\tilde{g}$ maps bijectively onto $(-\infty, c]$, there is a $\xi<\eta$ such that $(x-y)/t=\tilde{g}((\xi-\eta)/t)$. It may be that $V_0(\eta)>y$ if V_0 jumps at η, but for all $\eta' \in (\xi, \eta)$ it is true that $V_0(\eta') \leqslant y$. [It was for this reason that we chose $y>x-ct$ so that there is some room to reduce η. We do not want $\eta'<\xi$ because then $(\xi-\eta')/t$ would not be in the domain of $\tilde{g}$.] Then we have

$$\begin{aligned} x &= y+t\tilde{g}\left(\frac{\xi-\eta}{t}\right) \\ &\geqslant V_0(\eta')+t\tilde{g}\left(\frac{\xi-\eta'}{t}\right)+\varepsilon(\eta, \eta') \\ &\geqslant V(\xi, t)+\varepsilon(\eta, \eta') \end{aligned}$$

where we used Eq. (101), and $\varepsilon(\eta, \eta') \to 0$ as $\eta' \nearrow \eta$, by the uniform continuity of $\tilde{g}$. We can conclude that $V(\xi, t) \leqslant x$. Then by Eq. (102), by the choices of η and ξ, and by Lemma 9,

$$R(x, t) \geqslant \xi=R_0(y)+tj^*\left(\frac{x-y}{t}\right)$$

This is valid for all $y > x - ct$, and extends by continuity also to $y = x - ct$. We have proved one half of Eq. (105).

For the converse, let ξ_1 be strictly above the quantity on the right-hand side of Eq. (105). Let $\eta \geqslant \xi_1$ be arbitrary. Pick y so that $(x-y)/t = \tilde{g}((\xi_1 - \eta)/t)$. By Lemma 9 this is equivalent to $\xi_1 - \eta = tj^*((x-y)/t)$. By the choice of ξ_1,

$$\xi_1 > R_0(y) + tj^*((x-y)/t) = R_0(y) + \xi_1 - \eta$$

so that $R_0(y) < \eta$, which in turn implies that $V_0(\eta) \geqslant y$. Thus

$$V_0(\eta) + t\tilde{g}\left(\frac{\xi_1 - \eta}{t}\right) \geqslant y + x - y = x$$

Since this is valid for all $\eta \geqslant \xi_1$, Eq. (101) gives $V(\xi_1, t) \geqslant x$ and then Eq. (102) implies $R(x, t) \leqslant \xi_1$. We have proved Eq. (105), and thereby completed the proof of Theorem 3.

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